Lecture 25

Plan:

- Finsh §9.4 Linear system
- Start §9.5 Solving homogeneous Linear system.

§9.4. Linear system

Recall 1st order linear system is:

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$

Where $\vec{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $P(t) = [P_{ij}(t)]_{1 \le i,j \le n}$

coefficient matrix

$$\vec{f} = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

E.g.:

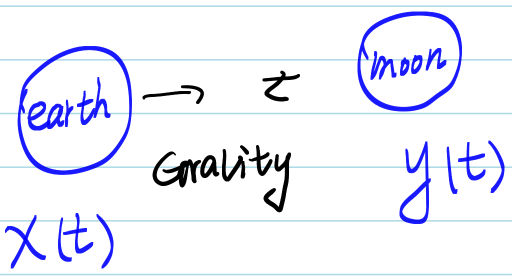
$$x_1' = 4x_1 - 3x_2$$
$$x_2' = 6x_1 - 7x_2$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\frac{d\vec{x}}{dt}} = \underbrace{\begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}}_{P(t)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

$n=2$

Remark: In practical problems, we often have to consider more than one D.Es.

E.g two body problem



$\ddot{x}(t) \sim$ gravity

$\ddot{y}(t) \sim$ gravity

Remark: Indeed, any high order D.E can be written as a 1st order linear system.

E.g. Write $x''' - tx' + x = \sin t$ as a 1st order linear system in matrix form.

A: Step 1: write $x_1 = x$, $x_2 = x'$, $\cdots$, $x_k = x^{(k-1)}$ where k is the order of the D.E

Write
$$\begin{cases} x_1 = x \\ x_2 = x' \\ x_3 = x'' \end{cases}$$

Step 2: Express each x_i' as a linear function of $x_1, \cdots, x_k$ and maybe a function of t.

$$\Rightarrow \quad x_1' = x_2 \qquad ①$$

$$x_2' = x_3 \qquad ②$$

Q: How about x_3'?

Since $x''' = tx' - x + \sin t$

$\Rightarrow \quad x_3' = tx_2 - x_1 + \sin t \quad ③$

Step 3: putting together ①, ②, ③.

$$\Rightarrow \begin{cases} x_1' = 0\cdot x_1 + x_2 + 0\cdot x_3 + 0 \\ x_2' = 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + 0 \\ x_3' = -x_1 + t x_2 + 0\cdot x_3 + \sin t \end{cases}$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}}_{\frac{d\vec{x}}{dt}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & t & 0 \end{bmatrix}}_{\substack{P(t) \\ 3\times 3}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\substack{\vec{x} \\ 3\times 1}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}}_{\vec{f}(t)}$$

" $x''' - tx' + x = \sin t$ " $\Rightarrow$

$$\Rightarrow \quad \text{“} \frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t) \text{”} \quad (*)$$

(Hence to solve " $x''' - tx' + x = \sin t$ "

$\Leftrightarrow$ to solve the 1st order linear system $(*)$)

Next we discuss how to solve 1st order linear system. We start with homogeneous linear system

Recall

$$\frac{d\vec{x}}{dt} = P(t)\vec{x}$$ ---- homogeneous linear system

$\downarrow$

39.5

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$ --- nonhomogeneous linear system

$\downarrow$

39.7

Recall in last Lecture 24, we considered

$$\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 6x_1 - 7x_2 \end{cases} \Leftrightarrow \underbrace{\frac{d\vec{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \vec{x}}_{\text{matrix form}}$$

We verified that

$$\vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

are both solutions to the system

We also verified that $c_1\vec{x}_1 + c_2\vec{x}_2$ is also a solution for any $c_1, c_2 \in \mathbb{R}$

In general, we have the following thm.

Thm (Homogeneous system)

Let $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ be n solns of the homogeneous linear system of

$$\frac{d\vec{x}}{dt} = P(t)\vec{x}$$

Then for $c_1, \cdots, c_n \in \mathbb{R}$,

$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$ is also a soln.

We can compare it with solving 2nd order linear D.E:

$$ay'' + by' + cy = 0$$

- Assume y_1, y_2 are both solns to the above. Then $c_1y_1 + c_2y_2$ is also a soln.

- If y_1, y_2 are two linearly independent solution. then the general solution is

$$y = c_1y_1 + c_2y_2$$

means every soln can be written in the form: $c_1y_1 + c_2y_2$

For a linear system, it is similar:

Thm (General soln of homogeneous system)

Let $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ be n linearly independent soln s to the homogeneous linear system

$\vec{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ $\quad \dfrac{d\vec{x}}{dt} = P(t)\vec{x}$, where $P(t)$ is an $n \times n$ matrix function

Then the general soln to the above linear system is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n ;$$

Here $c_1, c_2 \cdots c_n \in \mathbb{R}$.

Q: What does it mean by "$\vec{x}_1, \cdots, \vec{x}_n$ is linearly independent"?

A: See the following def^n.

Def^n: m vector functions $\vec{x}_1(t), \dots, \vec{x}_m(t)$ are said to linearly dependent on the interval I if there exist constants $c_1, \cdots, c_m$, not all equal to 0, such that

$$c_1 \vec{x}_1(t) + \cdots + c_m \vec{x}_m(t) = 0 .$$

for all $t \in I$.

Otherwise, we say $\vec{x}_1(t), \dots, \vec{x}_m(t)$ are linearly independent on I

An important criteria for linear independence:

Thm: Let $\vec{x}_1(t), \dots, \vec{x}_n(t)$ be n-dimensional vector functions on an interval I. Write

$$\vec{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \ \vec{x}_2 = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{bmatrix}, \dots, \vec{x}_n = \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}.$$

Then $\vec{x}_1, \dots, \vec{x}_n$ are linearly independent $\iff$

$$\det([\vec{x}_1(t), \cdots, \vec{x}_n(t)]) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ \underbrace{x_{n1}(t)}_{\vec{x}_1} & \underbrace{x_{n2}(t)}_{\vec{x}_2} & \cdots & \underbrace{x_{nn}(t)}_{\vec{x}_n} \end{vmatrix}_{n \times n} \neq 0$$

Notation: The above determinant is called the wronskian of $\vec{x}_1, \cdots, \vec{x}_n$:

$$W(\vec{x}_1, \cdots, \vec{x}_n) = \det([\vec{x}_1, \cdots, \vec{x}_n])$$

Eg1: Let $\vec{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$, $\vec{x}_2(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ -e^{2t} \end{bmatrix}$,

$\vec{x}_3(t) = \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ e^{2t} \end{bmatrix}$.

prove $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are linearly independent on $\underbrace{\mathbb{R} = (-\infty, +\infty)}_{I}$.

Pf: Note:

$$W(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \begin{vmatrix} e^{2t} & e^{2t} & e^{2t} \\ 0 & e^{2t} & 2e^{2t} \\ e^{2t} & -e^{2t} & e^{2t} \end{vmatrix}$$

$$= e^{2t} \begin{vmatrix} e^{2t} & 2e^{2t} \\ -e^{2t} & e^{2t} \end{vmatrix} - e^{2t} \begin{vmatrix} 0 & 2e^{2t} \\ e^{2t} & e^{2t} \end{vmatrix} + e^{2t} \begin{vmatrix} 0 & e^{2t} \\ e^{2t} & -e^{2t} \end{vmatrix}$$

$$= 4e^{6t} \neq 0$$

$\Rightarrow \vec{x}_1, \vec{x}_2, \vec{x}_3$ are linearly independent on $\mathbb{R}$.

E.g2: Let $\vec{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$, $\vec{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

prove $\vec{x}_1(t)$, $\vec{x}_2(t)$ are linearly independent on $\mathbb{R} = (-\infty, +\infty)$.

Pf: Note

$$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix}$$

$$= 7e^{-3t} \neq 0$$

Hence $\vec{x}_1$, $\vec{x}_2$ are linearly independent on $\mathbb{R}$.

Q: Find the general soln for the homogeneous linear system:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \vec{x}$$

A: In last lecture 24, we verified

$$\vec{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \qquad \vec{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

In the above E.g 2, we showed

$\vec{x}_1(t)$, $\vec{x}_2(t)$ are linearly independent on $\mathbb{R}$.

Recall for a homogeneous linear system

$$\frac{d\vec{x}}{dt} = \underbrace{P(t)}_{n\times n}\vec{x}$$

If $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ are linearly independent solns.

$\Rightarrow$ $x = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n$, $c_1, \cdots, c_n \in \mathbb{R}$,

gives the general soln.

Hence the general soln is

$$x = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

$$= c_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

$$= \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

Hence in general, to solve $\frac{d\vec{x}}{dt} = P(t)\vec{x}$

the key thing is to find n linearly independent solns

$$\vec{x}_1, \cdots, \vec{x}_n$$

(once this is done, the general soln is just $\vec{x} = c_1 \vec{x}_1 + \cdots + c_n \vec{x}_n$)

In next lecture 26, we will discuss how to find a fundamental soln set "$\{\vec{x}_1, \cdots, \vec{x}_n\}$" for

$$\frac{d\vec{x}}{dt} = P(t)\vec{x}$$

when $P(t) = A$ is an $n \times n$ constant matrix.

More precisely, in Lecture 26, we consider

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

← called homogeneous constant coefficients system

where $A = [a_{ij}]_{1 \le i,j \le n}$, $a_{ij} \in \mathbb{R}$.

E.g.:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \vec{x}$$

Recall we verified

$$e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = e^{-5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

are solutions

Inspaired by the above E.g. we try the test vector function $\vec{x} = e^{\lambda t}\vec{v}$ where $\lambda \in \mathbb{R}$, and $\vec{v}$ is a constant vector:

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \text{ to be decided.}$$

We plug into $\frac{d\vec{x}}{dt} = A\vec{x}$

LHS: Since $\vec{x} = e^{\lambda t}\vec{v} = \begin{bmatrix} e^{\lambda t}v_1 \\ \vdots \\ e^{\lambda t}v_n \end{bmatrix}$

$$\Rightarrow \quad \frac{d\vec{x}}{dt} = \begin{bmatrix} \lambda e^{\lambda t}v_1 \\ \vdots \\ \lambda e^{\lambda t}v_n \end{bmatrix} = \lambda e^{\lambda t}\vec{v}$$

RHS: $A\vec{x} = e^{\lambda t}A\vec{v}$

$$\text{LHS} = \text{RHS} \Rightarrow \lambda \cancel{e^{\lambda t}}\vec{v} = \cancel{e^{\lambda t}}A\vec{v}$$

$$\Rightarrow \lambda\vec{v} = A\vec{v}$$

of course. $\vec{v}=0$ is a soln.
But we don't want $\vec{x}=e^{\lambda t}\vec{v}$ to be 0.
why? we want "Linearly independent solutions $\vec{x}_1, \cdots, \vec{x}_n$"
By linear algebra,
if $\vec{x}_1, \cdots, \vec{x}_n$ are linearly independent
$\Rightarrow$ none of them can be zero

Now,

Q: Can we find $\lambda \in \mathbb{R}$, $\vec{v} \neq 0$ s.t

$$\lambda \vec{v} = A\vec{v}$$

$$\text{or} \quad \underset{n\times n}{A}\,\underset{n\times 1}{\vec{v}} = \lambda \vec{v} \qquad (*)$$

Recall from linear algebra,

Def^n: if $\lambda \in \mathbb{R}$ and $\vec{v} \neq 0$ satisfy $(*)$,

then λ —— eigenvalue of A

$\vec{v}$ —— eigenvector of A that is associated with λ.